

QUANTIZATION OF A PARTICLE IN A BACKGROUND YANG-MILLS FIELD

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Abstract

Two classes of observables defined on the phase space of a particle are quantized, and the effects of the Yang-Mills field are discussed in the context of geometric quantization.

PACS: 03.65.Bz, 02.40.Vh

Short title: Particle in background Yang-Mills field.

I. Introduction.

Let Q be a Riemannian manifold considered as the configuration space of a particle, the purpose of this paper is to discuss the quantization of the observables on the phase space T^*Q of this particle when it is moving under the influence of a background Yang-Mills field, the Yang-Mills potential is a connection α on a principal bundle N over Q .

The free G -action on N can be lifted to a Hamiltonian G -action on T^*N with an equivariant moment map $J : T^*N \rightarrow \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$, and denote by \mathcal{O}_μ the coadjoint orbit through μ . Then $J^{-1}(\mathcal{O}_\mu)/G$ has a canonical symplectic structure given by the Marsden-Weinstein reduction [1]. This reduced phase space is the appropriate phase space of a particle in a background Yang-Mills field α of charge μ [2].

We will denote by $\mathcal{Q}(X)$ the quantization of the phase space X , suppressing in our notation the choices of polarizations and pre-quantization line bundles *etc*, via the standard procedure of geometric quantization [3, 4]. Suppose we choose the vertical polarization on T^*N so that the quantization $\mathcal{Q}(T^*N)$ of T^*N gives $L^2(N)$. Moreover, suppose the co-adjoint orbit \mathcal{O}_μ is integral, so that the quantization of this coadjoint orbit gives a irreducible representation space \mathcal{H}_μ of G [5] [6]. A theorem of Guillemin-Sternberg [7] (see also [8]) then suggests that the quantization of $J^{-1}(\mathcal{O}_\mu)/G$ is given by $\text{Hom}_G(\mathcal{H}_\mu, L^2(N))$, the space of G -equivariant linear maps from \mathcal{H}_μ to $L^2(N)$. And this result holds independent of whether there is a Yang-Mills field present in the background. Thus when some technical assumptions are made so that the procedure of geometric quantization can be carried out smoothly, the Yang-Mills field plays no role in the quantization of the phase space $J^{-1}(\mathcal{O}_\mu)/G$.

We will discuss the effect of the Yang-Mills field in quantizing observables that are lifted from functions on T^*Q . We will show that the resulting quantum operators are expressed in terms of the covariant derivatives, which is defined by the connection α . In particular, we will show that the quantum operators for f of the form $\frac{1}{2}||p||^2 + V(q)$ are expressed in terms of the covariant Laplace operator and the Ricci curvature. This is obtained by a standard Blattner-Kostant-Sternberg (BKS) pairing approach [9]. Our results are in agreement with those of Landsman [10] who arrived at the conclusion via deformation quantization.

Outline of this paper is as follows; In section 2 we give a detailed exposition of our problem in order to standardize the notations used throughout this paper. As a prelude to our result, we note that if the gauge group is abelian, we will recover the Dirac quantization of a charge particle in the presense of an electro-magnetic field. In section 3 we introduce local coordinates to facilitate our calculation, and state some results concerning the Hamiltonian vector fields for our observables. We follow closely the treatment of [8] on the polarization chosen for the phase space in section 4. Our results when f is polarization preserving are given in section 5, and the quantization of $\frac{1}{2}||p||^2 + V(q)$ using BKS pairing appears in section 6.

II. Preliminary discussions.

Let N be a principal G -bundle over Q where G is compact, with Lie algebra \mathfrak{g} , and the group action is on the right. We define two functions $R_g : N \rightarrow N$ and $\hat{n} : G \rightarrow N$, where

$$R_g(n) = \hat{n}(g) = ng, \quad (1)$$

and we denote by E the Jacobian of E . A connection is a linear map $\alpha(x) : T_x N \rightarrow \mathfrak{g}$

\mathfrak{g} for each $n \in N$ satisfying

- i. $\text{Ad}_{g^{-1}}\alpha(n) = \alpha(ng)R_{g^*} : T_n N \rightarrow \mathfrak{g}$,
- ii. $\alpha(n)\hat{n}_* = \text{Id} : \mathfrak{g} \rightarrow \mathfrak{g}$.

The free G -action on N can be lifted to a Hamiltonian G -action on T^*N with moment map $J : T^*N \rightarrow \mathfrak{g}^*$ given by $J(\xi, n) = \hat{n}^* \cdot \xi$.

Let $N^\#$ be the pullback bundle over T^*Q : Explicitly,

$$N^\# = \{(p, n) \mid p \in T_q^*Q \text{ where } q = \pi(n)\}.$$

Define a diffeomorphism

$$\begin{aligned} \chi : N^\# \times \mathfrak{g}^* &\rightarrow T^*N, \\ (p, n, \mu) &\mapsto (\xi, n) \quad \text{where} \quad \xi = \pi^*(p) + \langle \mu, \alpha \rangle \in T_n^*N. \end{aligned} \quad (2)$$

This map in turn induces an α -dependent projection $\pi_\alpha : T^*N \rightarrow T^*Q$, and their corresponding symplectic forms are related by

$$\Omega_{T^*N} = \pi_\alpha^* \Omega_{T^*Q} + d\langle \mu, \alpha \rangle. \quad (3)$$

One shows that the moment map is simply the projection $N^\# \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$, *i.e.*, $\chi^{-1}(J^{-1}(\mu)) = N^\# \times \{\mu\}$. Thus for each μ , the G -action on $N^\#$ induces a G -action on $J^{-1}(\mu)$. This action is non-canonical ($R_g^* \Omega_{T^*N} \neq \Omega_{T^*N}$) in general:

$$(\xi, n) \mapsto (\xi_g, ng) \quad \text{where} \quad \xi_g = R_{g^{-1}}^* \xi + R_{g^{-1}}^*[(\text{Ad}_{g^{-1}}^* - \text{Id})\mu]\alpha(n).$$

However, this action coincides with the canonical G -action when restricted to the isotropy subgroup

$$H = \{g \in G \mid \text{Ad}_g^* \mu = \mu\}$$

of μ . (There $\text{Ad}_{g^{-1}}^* \mu = \mu$ and $\xi_g = R_{g^{-1}}^* \xi$.) The relevant phase space becomes $J^{-1}(\mathcal{O}_\mu)/G = J^{-1}(\mu)/H = N^\# / H \times \{\mu\}$.

Given an observable on the phase space of the particle $f : T^*Q \rightarrow \mathbf{R}$, by the projection π_α , the pullback map, which we continue to denote by f ,

$$f = \pi_\alpha^* f : T^*N \simeq N^\# \times \mathfrak{g}^* \longrightarrow \mathbf{R}, \quad (4)$$

is invariant with respect to both the canonical G -action and the non-canonical one. In particular, f is independent on the charge variables $\mu \in \mathfrak{g}^*$.

We assume that \mathcal{O}_μ is integral, so that $\mathcal{Q}(\mathcal{O}_\mu) = \mathcal{H}_\mu$ is an irreducible representation space of G induced by $\rho_\mu : H \rightarrow U(1)$. Choose an orthonormal basis $\{\phi_i\}$ for \mathcal{H}_μ , where ϕ_i is a holomorphic function on the Kähler manifold G/H [5]. Then $\Psi \in \mathcal{Q}(J^{-1}(\mu)/H) = \text{Hom}_G(\mathcal{H}_\mu, L^2(N))$ is determined by $\Psi(\phi_i) = \Psi_i \in L^2(N)$. Using orthonormality of ϕ_i and G -equivariance of Ψ , we write

$$\Psi = \sum \Psi_i \phi_i : N \times G/H \rightarrow N \times_G G/H \rightarrow \mathbf{C}$$

which is uniquely determined by $\psi = \Psi(-, eH) : N \rightarrow \mathbf{C}$ with the condition $\psi(hx) = \rho(h^{-1})\psi(x)$ for all $h \in H$. So we see that the Yang-Mills potential plays

no role in quantizing the relevant phase space, it simply picks up the multiplicity of the charge sector \mathcal{O}_μ in $L^2(N)$ (cf. [7], [8]).

If we are to quantize an observable that is a pullback of f on the phase space of the particle T^*Q , the connection α plays an important role. As an illustration, suppose the charge $\mu \in \mathfrak{g}^*$ is G -invariant, then $H = G$. This will be the case if for instance G is abelian. Under this condition, $J^{-1}(\mu)/H$ is diffeomorphic to T^*Q via the projection π_α , and the canonical symplectic form on the reduction space pushes forward onto T^*Q . So $J^{-1}(\mu)/H$ is symplectomorphic to T^*Q if T^*Q is equipped with the “effective” symplectic form $\Omega_{\text{eff}} = \Omega_{T^*Q} + \langle \mu, \Omega_\nabla \rangle$ where Ω_∇ is a two-form on Q which pulls back to the curvature form $d\alpha$ on N [11]. It is with respect to this effective symplectic form that the quantization procedure must be carried out. Since the adjustment is a two form on Q , quantization of T^*Q using the vertical polarization still gives $L^2(Q)$. However, Hamiltonian vector fields \mathcal{H}_{f_i} associated with observables f_i and the Poisson bracket are defined in terms of the effective form:

$$\Omega_{\text{eff}}(\mathcal{H}_{f_i}, -) = -df, \quad \{f_1, f_2\} = \Omega_{\text{eff}}(\mathcal{H}_{f_1}, \mathcal{H}_{f_2}).$$

The quantization of observables must preserve Poisson bracket

$$[\mathcal{Q}(f_1), \mathcal{Q}(f_2)] = i\hbar \mathcal{Q}(\{f_1, f_2\}).$$

When carry out the geometric quantization with respect to Ω_{eff} , the result is the Dirac quantization with α as the vector potential associated with a electro-magnetic field (cf. [3]).

III. The Hamiltonian vector fields.

Let us first introduce local canonical coordinates (ξ_i, n_i) on T^*N so that $\Omega_{T^*N} = d\xi_i dn_i$, similarly (p_a, q_a) on T^*Q with $\Omega_{T^*Q} = dp_a dq_a$, and let $\alpha = A_{si} dn_i$ where $i = 1 \dots \dim N$, $a = 1 \dots \dim Q$, $s = 1 \dots \dim G$, and repeated indices are summed. For each $n \in N$, let us denote the horizontal lift $T_q Q \rightarrow T_n N$ by the matrix $M_{i\sigma}(n)$, $i = 1 \dots \dim N$, $\sigma = 1 \dots \dim Q$. We have

$$\frac{\partial q_a}{\partial n_i} M_{ib} = \delta_{ab} \quad (5)$$

and the covariant derivative is the horizontal lift of $\frac{\partial}{\partial q_a}$:

$$\mathcal{D}_a = M_{ia} \frac{\partial}{\partial n_i}. \quad (6)$$

In these coordinates, the canonical one-form and the symplectic two-form of T^*N can be calculated using (3)

$$\begin{aligned} \chi^* \xi_i dn_i &= \left(p_a \frac{\partial q_a}{\partial n_i} + \mu_s A_{si} \right) dn_i \\ \chi^* \Omega_{T^*N} &= \frac{\partial q_a}{\partial n_i} dp_a dn_i + \mu_s \frac{\partial A_{si}}{\partial n_j} dn_j dn_i + A_{si} d\mu_s dn_i. \end{aligned} \quad (7)$$

Let $f : N^\# \times \mathfrak{g}^* \rightarrow \mathbf{R}$ be a pullback function from T^*Q , and let \mathcal{H}_f be its Hamiltonian vector field

$$\mathcal{H}_f = B_a \frac{\partial}{\partial \mu_a} + C_i \frac{\partial}{\partial n_i} + U_s \frac{\partial}{\partial \mu_s}.$$

Using $\chi^* \Omega_{T^*N}(\mathcal{H}_f, -) = -df$ we get

$$\begin{aligned} \frac{\partial f}{\partial q_a} \frac{\partial q_a}{\partial n_i} &= -B_a \frac{\partial q_a}{\partial n_i} + \mu_s \left(\frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) C_j - A_{si} U_s, \\ \frac{\partial f}{\partial p_a} &= \frac{\partial q_a}{\partial n_i} C_i, \\ \frac{\partial f}{\partial \mu_s} &= A_{si} C_i. \end{aligned} \quad (8)$$

Since f is invariant with respect to the canonical G -action, \mathcal{H}_f is tangent to the subspace $J^{-1}(\mu) \simeq N^\# \times \{\mu\}$, thus $U_s = 0$. As remarked after (4), f is independent of μ , thus $A_{si} C_i = 0$, which implies \mathcal{H}_f is horizontal. Moreover, letting $B_a = -\frac{\partial f}{\partial q_a} + E_a$, we have

$$\mathcal{H}_f = \left[-\frac{\partial f}{\partial q_a} \frac{\partial}{\partial p_a} + C_i \frac{\partial}{\partial n_i} \right] + E_a \frac{\partial}{\partial p_a}.$$

The terms in the bracket is the horizontal lift of the Hamiltonian vector field of f with respect to the usual symplectic form Ω_{T^*Q} on T^*Q . E_a satisfies

$$E_a \frac{\partial q_a}{\partial n_i} = \mu_s \left(\frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) C_j.$$

So we summarize the properties of \mathcal{H}_f needed for our purpose.

Proposition 1. *If $f : T^*N \rightarrow \mathbf{R}$ is a pullback of a function on T^*Q , then for all $\mu \in \mathfrak{g}$ the Hamiltonian vector field \mathcal{H}_f is a vector field on the subspace $J^{-1}(\mu)$, and as the total space of a principal bundle over T^*Q with connection α , \mathcal{H}_f is horizontal. It differs from the horizontal lift of the standard Hamiltonian vector field on T^*Q by a field in the vertical direction. With respect to the local coordinates chosen, we have explicitly:*

$$\mathcal{H}_f = \left[-\frac{\partial f}{\partial q_a} \frac{\partial}{\partial p_a} + M_{ia} \frac{\partial f}{\partial p_a} \frac{\partial}{\partial n_i} \right] + \mu_s M_{ia} M_{jb} \left(\frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) \frac{\partial f}{\partial p_b} \frac{\partial}{\partial p_a}. \quad (9)$$

Furthermore, we have

$$\langle \chi^* \xi_i dn_i, \mathcal{H}_f \rangle = p_a \frac{\partial f}{\partial p_a}. \quad (10)$$

IV. Polarization.

We first state some well known results concerning the quantization of integral coadjoint orbits \mathcal{O}_μ . Let \mathfrak{h} be the Lie algebra of H , we say that \mathcal{O}_μ is integral if the map

$$v \in \mathfrak{h} \rightarrow 2\pi i \langle v, \mu \rangle \quad (11)$$

is the derivative of a global character, *i.e.*, there is a group homomorphism $\rho_\mu : H \rightarrow U(1)$ such that $\rho_{\mu*}$ is the map given in (11). A version of the Borel-Weil theorem, due to Kirillov [5] and Kostant [6] asserts that there is a one-to-one correspondence between the integral orbits of G and its unitary irreducible representations, and

these representations can be construction by the method of geometric quantization applied to the coadjoint orbit \mathcal{O}_μ , which we will briefly explain.

Let \mathcal{L} be the prequantization line bundle $G \times_{\rho_\mu} \mathbf{C}$ over $\mathcal{O}_\mu \simeq G/H$ with connection induced by the map ρ_μ . It is known that \mathcal{O}_μ is a Kähler manifold with complex coordinates with respect to which the G -action is holomorphic. There is a standard G -equivariant polarization quantizing with respect to the line bundle and this polarization gives \mathcal{H}_μ whose elements are holomorphic functions on \mathcal{O}_μ . The polarization, known as the positive Kähler polarization, is given by left translation of a set of $v_k \in \mathfrak{g} \otimes \mathbf{C}$, the complexification of \mathfrak{g} , so that the polarization is generated by

$$V_k(\text{Ad}_g \mu) = g_* v_k \in T_{\text{Ad}_g \mu} \mathcal{O}_\mu. \quad (12)$$

As a polarization, v_k thought of as vectors in $T_\mu \mathcal{O}_\mu$ satisfies

$$\Omega_{\mathcal{O}_\mu}(v_h, v_k) = 0 \quad (13)$$

where $\Omega_{\mathcal{O}_\mu}$ is the canonical symplectic form on \mathcal{O}_μ . The specifics of the choices of v_k will not be important in what follows. It is worth mentioning that V_k is contained in the vertical polarization on T^*N .

Consider the complex distribution on $N^\# \times \mathcal{O}_\mu$ generated by $\{\frac{\partial}{\partial p_a}, V_k\}$, it is G -equivariant thus projects onto $N^\# \times_G \mathcal{O}_\mu \simeq N^\# / H \times \{\mu\}$. One checks that the image is a polarization which we denote by \mathcal{P} [8].

It is easy to represent \mathcal{P} in local coordinates on $N^\#$; $\frac{\partial}{\partial p_a}$ are vector fields on $N^\#$, and V_k corresponds to $\hat{n}_* v_k$ as complex vector fields on N , \hat{n}_* as in (1). This is so since the assignment $(p, n, \text{Ad}_g \mu) \mapsto (p, ng, \mu)$ defines the projection $N^\# \times \mathcal{O}_\mu \rightarrow N^\# \times_G \mathcal{O}_\mu \simeq N^\# / H \times \{\mu\}$. Thus vector field generated by G -action on \mathcal{O}_μ translates to vector field generated by G -action on N .

The Hilbert space structure on $L^2(N)$ is given by integration with respect to the measure $dn = d\sigma \sqrt{\det(g)} dq$, where $d\sigma$ is a Haar measure on G which we transfer to a measure on the fiber in the projection $N \rightarrow Q$ and g is the metric on Q . Using the half-form bundle formalism the wavefunctions are of the form $\psi(n) \sqrt{dn}$. It is clear the the Haar measure will play no role in our consideration as the polarization and all Hamiltonians in question are G -invariant. To keep the half-form bundle formalism to a minimum, we may identify the wavefunctions as $\psi(n) \det g^{1/4}$. We will determine explicitly the differential operators corresponding to f so that

$$\psi(n) \det g^{1/4} \mapsto [\mathcal{Q}(f) \psi(n)] \det g^{1/4}. \quad (14)$$

In quantizing f that is linear in the momentum variables, the $\det g^{1/4}$ term will give rise to the covariant divergence, and for $f = \frac{1}{2} \|p\|^2 + V(q)$, it results in the Ricci curvature. The appearance of the Ricci curvature is also reported in [4].

V. Polarization preserving case.

If $f : T^*Q \rightarrow \mathbf{R}$ is linear in p , $f = K_a(q)p_a$, then one easily checks, using (9), that $\exp t \mathcal{H}_f \mathcal{P} = \mathcal{P}$. In fact, we have

Proposition 2.

$$\left[\mathcal{H}_f, \frac{\partial}{\partial p_a} \right] = \frac{\partial K_a}{\partial q_b} \frac{\partial}{\partial p_b}, \quad (15)$$

$$[\mathcal{H}_f, V_k] \in \text{span} \left\{ \frac{\partial}{\partial p_a} \right\}, \quad (16)$$

where the brackets refer to the Lie algebra bracket on vector fields.

Proof. Equation (15) is by direct computation. We have

$$\mathcal{H}_f = \left(E_b - p_c \frac{\partial K_c}{\partial q_b} \right) \frac{\partial}{\partial p_b} + K_b M_{ib} \frac{\partial}{\partial n_i} \quad (17)$$

where E_b is independent of p_a , and (15) results.

To show (16), we first realize from Proposition 1 that $\mathcal{H}_f = W_1 + W_2$ where W_1 is the horizontal lift of a vector field on T^*Q , thus $[W_1, V_k] = 0$ as V_k is generated by the group action on N . W_2 is of the form $F_a \frac{\partial}{\partial p_a}$. Since V_k is independent of p , $[W_2, V_k] = V_k(F_a) \frac{\partial}{\partial p_a}$, where $V_k(F_a)$ refers to applying the vector field as a differential operator to the coefficient function F_a . \square

The importance of (16) is that $[\mathcal{H}_f, V_k]$ is a combination of vectors fields in \mathcal{P} which does not involve the V_h vector fields. According to (7.12) of [3], the quantization of f is then given by:

$$\begin{aligned} \mathcal{Q}(f)\psi &= -\frac{i\hbar}{\det g^{1/4}} \left[\mathcal{H}_f(\psi(n) \det g^{1/4}) + \frac{1}{2} \sum_{a=1}^{\dim Q} \frac{\partial K_a}{\partial q_a} (\psi(n) \det g^{1/4}) \right] \\ &= -i\hbar \left[\mathcal{H}_f(\psi) + \frac{1}{2} \left(\frac{1}{\sqrt{\det g}} \mathcal{H}_f(\sqrt{\det g}) + \sum_{a=1}^{\dim Q} \frac{\partial K_a}{\partial q_a} \right) \psi \right] \end{aligned} \quad (18)$$

The Hamiltonian vector field \mathcal{H}_f projects to a vector field $V^\#$ on N , which is the horizontal lift to the projection of \mathcal{H}_f onto Q where $V = K_b \frac{\partial}{\partial q_b}$. Note that $\mathcal{H}_f(\sqrt{\det g}) = V(\sqrt{\det g})$. The divergence of the vector field V on Q is defined [12] through the relation

$$d * V = \operatorname{div} V \sqrt{\det g} dq$$

The covariant divergence on N is defined as the divergence of the horizontal lift $V^\#$. We have

$$\operatorname{div} V = \frac{1}{\sqrt{\det g}} V(\sqrt{\det g}) + \sum_{a=1}^{\dim Q} \frac{\partial K_a}{\partial q_a}.$$

Using (17), (18) and the fact that ψ is independent of p , we have

Proposition 3. $\mathcal{Q}(f)\psi = -i\hbar(K_a \mathcal{D}_a + \frac{1}{2} \operatorname{div} V)\psi$ where \mathcal{D} is the covariant derivative with respect to the connection α .

Since $\det g$ is a function of n through q , the divergence and the covariant divergence are the same.

VI. BKS pairing case.

Let \mathcal{P} and \mathcal{P}' be transversal polarizations on T^*N , denote their associated quantum spaces by \mathcal{Q} and \mathcal{Q}' . The BKS pairing gives rise to a map $B : \mathcal{Q}' \rightarrow \mathcal{Q}$ such that

$$\langle B(\psi), \phi \rangle = \int \psi \bar{\phi} (\det \omega)^{1/2} d\ell$$

where $d\ell$ is the Liouville form $d\xi_1 \dots d\xi_n dn_1 \dots dn_n$. Since the volume form on N is $dn = \sqrt{\det g} dn_1 \dots dn_n$, we have

$$B\psi(n) = \frac{1}{\sqrt{\det g}} \int_{T_n^* N} \psi(\det \omega)^{1/2} d\xi_1 \dots d\xi_n \quad (19)$$

Let $f = \frac{1}{2}g^{ab}p_ap_b + V(q)$, where g^{ab} is the inverse of the metric g_{ab} . From (9) we have

$$\begin{aligned} \mathcal{H}_f = & - \left(\frac{\partial V}{\partial q_a} + \frac{1}{2} \frac{\partial g^{bc}}{\partial q_a} p_b p_c \right) \frac{\partial}{\partial p_a} + M_{ia} g^{ab} p_b \frac{\partial}{\partial n_i} \\ & + \mu_s M_{ia} M_{jb} \left(\frac{\partial A_{sj}}{\partial n_i} - \frac{\partial A_{si}}{\partial n_j} \right) g^{bc} p_c \frac{\partial}{\partial p_a}, \end{aligned}$$

and note the linear dependence of the coefficients on the p variables. We denote by $\mathcal{P}_t = \exp t\mathcal{H}_f \mathcal{P}$, here the two polarizations \mathcal{P} and \mathcal{P}_t do not intersect transversely. We claim

Proposition 4. *Vector fields generated by the group action are in $\mathcal{P} \cap \mathcal{P}_t$.*

Proof. It suffices to show that

$$\Omega(\exp t\mathcal{H}_f V_k, \frac{\partial}{\partial p_a}) = 0 \quad (20)$$

$$\text{and} \quad \Omega(\exp t\mathcal{H}_f V_k, V_h) = 0 \quad (21)$$

with Ω as in (7). Let $\bar{p}(p, n, t)$ and $\bar{n}(p, n, t)$ denote the flow generated by $\exp t\mathcal{H}_f$ at $(p, n) \in N^\#$ with μ fixed, $(\bar{p}, \bar{n}, \mu) = \exp t\mathcal{H}_f(p, n, \mu)$. Then $\exp t\mathcal{H}_f V_k = V_k(\bar{p}_a) \frac{\partial}{\partial p_a} + V_k(\bar{n}_i) \frac{\partial}{\partial n_i}$. So $\Omega(\exp t\mathcal{H}_f V_k, \frac{\partial}{\partial p_a}) = \frac{\partial q_a}{\partial n_i} V_k(\bar{n}_i)$. Recall $V_k = \hat{n}_* v_k$ is vertical and $\frac{\partial q_a}{\partial n_i}$ is the Jacobian of the projection $\pi : N \rightarrow Q$. Then $\pi_* \hat{n}_* = 0$ implies (20) holds.

Equation (21) follows from general principle; Since f is G -invariant (with respect to the non-canonical G -action), V_h is equivariant with respect to the flow: $\exp t\mathcal{H}_f V_h(p, n) = V_h(\exp t\mathcal{H}_f(p, n))$. Since the flow of a Hamiltonian vector field preserves the symplectic form, we have

$$\Omega(\exp t\mathcal{H}_f V_k(p, n), V_h(\exp t\mathcal{H}_f(p, n))) = \Omega(V_k, V_h) = \Omega_{\mathcal{O}_\mu}(v_h, v_k) = 0,$$

where v_h and v_k belongs to a polarization on \mathcal{O}_μ to begin with (13). \square

This being the case, quantization of f via BKS pairing involves integrating only over the p variables, *i.e.*, the fiber coordinates of the projection $\Pi : N^\# \rightarrow N$. According to (7.20) of [3], together with the similarity transform (14) adjustment and the adjustment in the BKS pairing described in (19),

$$\mathcal{Q}(f)\psi(n) = \frac{1}{\det g^{1/2}} \frac{1}{\det g^{1/4}} i\hbar \frac{d}{dt} \Big|_{t=0} \Psi_t(n) \quad (22)$$

$$\text{where} \quad \Psi_t(n) = (i\hbar)^{-\dim Q/2} \int_{\Pi^{-1}(n)} [\det \omega_{ab}]^{1/2} \exp(i\hbar^{-1}L) \Psi(p, n, t) dp, \quad (23)$$

$$\Psi(p, n, t) = \psi(\bar{n}(p, n, t)) \times [\det g(\bar{n}(p, n, t))]^{1/4}, \quad (24)$$

$$\omega_{ab} = \Omega \left(\frac{\partial}{\partial p_a}, \exp t\mathcal{H}_f \frac{\partial}{\partial p_b} \right), \quad (25)$$

$$L = t \left(\frac{1}{2} \|p\|^2 + V(q) \right) - 2 \int_0^t V(\bar{n}(p, n, s)) ds. \quad (26)$$

The manipulation follows closely that of Sniatycki [3]. Making the substitution $x_a = tp_a$, we have results analogues to (7.26) and (7.27) of [3]:

Proposition 5.

$$\lim_{t \rightarrow 0+} t^{-\dim Q/2} \exp\left(\frac{i}{\hbar} \frac{\|x\|^2}{2t}\right) = (2\pi\hbar)^{\dim Q/2} e^{\pi i \operatorname{sgn}(g)/4} \sqrt{\det g} \delta(x). \quad (27)$$

$$\frac{\partial}{\partial t} t^{-\dim Q/2} \exp\left(\frac{i}{\hbar} \frac{\|x\|^2}{2t}\right) = \frac{i\hbar}{2} g_{ab} \frac{\partial^2}{\partial x_a \partial x_b} t^{-\dim Q/2} \exp\left(\frac{i}{\hbar} \frac{\|x\|^2}{2t}\right). \quad (28)$$

Proof. The first equation follows from the method of stationary phase (cf. [13]), which for n -dimensional space reads

$$\int_{\mathbf{R}^n} a(y) e^{ik\phi(y)} dy = \left(\frac{2\pi}{k}\right)^{n/2} \sum_{y|d\phi(y)=0} e^{\pi i \operatorname{sgn} H(y)/4} \frac{e^{ik\phi(y)} a(y)}{\sqrt{|\det H(y)|}} + O(k^{-n/2-1}).$$

The $\frac{1}{t}$ factor in (27) plays the role of the large parameter k . H is the Hessian of ϕ which in our case is $g^{\mu\nu}$, thus $\det H = \det g^{-1}$ and sgn is the signature of the metric. The only stationary point in (27) is $x = 0$, thus the right hand side of (27) has a $(\dim Q)$ -dimensional delta function at $x = 0$.

The second equation is a straight forward computation. We need the fact that $g_{a\mu} g^{\mu b} = \delta_{ab}$, and $\sum_a \sum_b g_{ab} g^{ab} = \sum_a \delta_{aa} = \dim Q$ in the course of the computation. \square

One checks that ω_{ab} in (29) is

$$\frac{\partial q_a}{\partial n_i} \frac{\partial \bar{n}_i}{\partial p_b} = t g^{ab} + \text{higher order terms in } t$$

using (30) below. Then $[\det \omega_{ab}]^{1/2} dp \sim t^{-\dim Q/2} dx$, providing us with the needed factor to apply the results of Proposition 5. Thus in determining $\frac{d}{dt}|_{t=0} \Psi_t(n)$, we need only to consider terms involving t , tp_a and $t^2 p_a p_b$ while ignoring terms of the form $t^2 p_a$ and all higher order terms. Using (9), the expansion of \bar{n} , expressed in the t and x_a variables, up to the relevant terms are

$$\begin{aligned} \tilde{n}_i(n, x) = & n_i + M_{ia} g^{\mu a} x_\mu + \frac{1}{2} \left[M_{jb} \left(\frac{\partial M_{ia}}{\partial n_j} g^{\nu b} g^{\mu a} \right. \right. \\ & \left. \left. + M_{ia} \frac{\partial g^{\mu a}}{\partial q_b} g^{\nu b} \right) - \frac{1}{2} M_{ia} \frac{\partial g^{\mu\nu}}{\partial q_b} g^{ba} \right] x_\mu x_\nu \end{aligned} \quad (30)$$

which is independent of t . And $\Psi_t(n)$ in (23) is reduced to

$$\Psi_t(n) = (i\hbar)^{-\dim Q/2} \int_{\Pi^{-1}(n)} t^{-\dim Q/2} \exp\left(\frac{i}{\hbar} \frac{\|x\|^2}{2t}\right) \Phi(n, x, t) dx,$$

$$\text{where } \Phi(n, x, t) = \exp(-i\hbar^{-1} t V(q)) \times \psi(\tilde{n}(n, x)) \det g(\tilde{n}(n, x))^{1/4}.$$

By applying Proposition 5, integration by parts yields

$$\mathcal{Q}(f)\psi(n) = \frac{(-2\pi i)^{\dim Q/2} e^{\pi i \operatorname{sgn} g/4}}{\det g^{1/4}} \left[-\frac{\hbar^2}{2} g_{ab} \frac{\partial^2 \Phi}{\partial x_a \partial x_b} \Big|_{x,t=0} + i\hbar \frac{d}{dt} \Big|_{x=t=0} \Phi \right] \quad (32)$$

The $\sqrt{\det H}$ term that appears in the method of stationary phase formula results in a $g^{1/2}$ factor (27) that cancels with the $\det g^{1/2}$ on the right hand side of (22).

Since n is fixed, we can choose a normal coordinate system [12] around $q = \pi(n)$ so that $\frac{\partial g^{\mu\nu}}{\partial q_a} = 0$ for all μ, ν and q_a when evaluated at n , (*i.e.*, at $x = t = 0$). A direct computation shows

$$\frac{d}{dt} \Big|_{x=t=0} \Phi(n, x, t) = -i\hbar^{-1} V(q) \psi(n) \det g^{1/4} \quad (33)$$

$$\begin{aligned} g_{ab} \frac{\partial^2}{\partial x_a \partial x_b} \Big|_{x=t=0} \Phi(n, x, t) &= \det g^{1/4} \left[g^{\mu\nu} M_{j\nu} \frac{\partial M_{i\mu}}{\partial n_j} \frac{\partial \psi}{\partial n_i} \right. \\ &\quad \left. + g^{\mu\nu} M_{i\mu} M_{j\nu} \frac{\partial^2 \psi}{\partial n_i \partial n_j} + \frac{1}{4} g^{\mu\nu} g^{ab} \frac{\partial^2 g_{\mu\nu}}{\partial q_a \partial q_b} \right]. \end{aligned} \quad (34)$$

Here we have made repeated use of the identity [12 p.302],

$$\frac{1}{\det g} \frac{\partial \det g}{\partial q_b} = \frac{1}{\det g} \frac{\partial \det g}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial q_b} = g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial q_b}.$$

In normal coordinates, the covariant Laplace operator reduces to

$$\begin{aligned} \Delta_\alpha \psi &= \frac{1}{\det g^{1/2}} \mathcal{D}_\mu (\det g^{1/2} g^{\mu\nu} \mathcal{D}_\nu \psi) \\ &= g^{\mu\nu} M_{i\mu} M_{j\nu} \frac{\partial^2 \psi}{\partial n_i \partial n_j} + g^{\mu\nu} M_{j\nu} \frac{\partial M_{i\mu}}{\partial n_j}, \end{aligned} \quad (35)$$

and the Ricci curvature becomes

$$\begin{aligned} R &= g^{ik} (\partial_k \Gamma_{ji}^j - \partial_j \Gamma_{ki}^j + \Gamma_{km}^j \Gamma_{ji}^m - \Gamma_{jm}^j \Gamma_{ki}^m) \\ &= (g^{ik} g^{\mu\nu} - g^{i\nu} g^{k\mu}) \partial_i \partial_k g_{\mu\nu} \\ &= \frac{3}{2} g^{ik} g^{\mu\nu} \partial_i \partial_k g_{\mu\nu} \end{aligned} \quad (36)$$

Here Γ_{ij}^k are the Christoffel symbols, and the identity

$$\Gamma_{ij,k}^m + \Gamma_{jk,i}^m + \Gamma_{ki,j}^m = 0$$

is used to show $g^{i\nu} g^{k\mu} \partial_i \partial_k g_{\mu\nu} = -\frac{1}{2} g^{ik} g^{\mu\nu} \partial_i \partial_k g_{\mu\nu}$. We must caution the readers that these expressions only hold in normal coordinates. However, by combining (32–36) we can express our final result in an coordinate invariant form:

Proposition 6. *Quantization of $f = \frac{1}{2} \|p\|^2 + V(q)$ gives*

$$\mathcal{Q}(f)\psi(n) = (-2\pi i)^{\dim Q/2} e^{\pi i \operatorname{sgn}(g)/4} \left(-\frac{\hbar^2}{2} \left[\Delta_\alpha + \frac{1}{6} R \right] + V(q) \right) \psi(n).$$

We conclude with a final remark. The Yang-Mills field is defined [14] as the curvature $\mathcal{D}\alpha$ of the Yang-Mills potential α , whereas the contribution of this connection in the local expression of the symplectic form is $d\alpha$. They are related by

$$\mathcal{D}\alpha(v, w) = d\alpha(\operatorname{hor} v, \operatorname{hor} w)$$

where hor denotes the horizontal projection. Since the vector fields of concern are all horizontal, the effect of $d\alpha$ is equivalent to the curvature

Acknowledgement. We wish to thank N.P. Landsman for helpful comments on the preliminary version of this work.

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